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ON THE REGULARITY OF ONE-DIMENSIONAL ELASTIC WAVES IN AN INCOMPRESSIBLE ISOTROPIC MATERIAL*

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Wave motion is considered in an incompressible isotropic material which is defined by quasi-linear hyperbolic system of four equations for which two characteristic fields are linearly degenerate in the sense of Lax. For the remaining characteristic fields a singificant nonlinearity is assumed. The behavior of derivatives of solution is investigated along these two different types of characteristic fields. The effect of the system nonlinearity shows itself in the unboundedness of derivatives with the limited solution along the essentially nonlinear fields.

In the linear theory of elasticity the smoothness of input data implies the stability of respective solutions. In nonlinear elasticity this effect is absent /1/.

It is shown in /2/ that when the system (of equations), which describes the motion of the system, is truly hyperbolic in the narrow sense, and is actually nonlinear /3/ and in addition the input data have a compact carrier and a fairly small C^2 norm, then the solution becomes infinite in finite time. An example is constructed in /2/ of a classic material with the quadratic function of the deformation energy, which is truly nonlinear, when the wave front does not contain the main deformation direction. The question is posed of the validity of these results in dependence of the true nonlinearity for one-dimensional elastic waves. A wide class of materials exists, which are not truly nonlinear. To these belong the isotropic incompressible materials, considered below.

1. The plane problem. The plane wave motion in incompressible elastic materials is defined by the system of equations /4,5/

$$\frac{\partial^2\Sigma}{\partial p_\alpha \partial p_1}\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2\Sigma}{\partial p_\alpha \partial p_2}\frac{\partial^2 u_2}{\partial x^2}=\rho_0\frac{\partial^2 u_\alpha}{\partial t^2},\quad \alpha=1,2 \tag{1.1}$$

where u_α are displacements, $p_\alpha=\partial u_\alpha/\partial x$ are gradients of deformation, and ρ_0 is the constant density in the undeformed state. For an isotropic material the function Σ of deformation energy depends only on invariants of the deformation tensor for which

$$I=II=3+p_1^2+p_2^2=3+q^2,\quad III\equiv 0$$

We shall consider the classical solutions of system (1.1), replacing it by the equivalent system of the first order

$$\frac{\partial U}{\partial t}+A(U)\frac{\partial U}{\partial x}=0,\ v_1=\frac{\partial u_1}{\partial t},\ v_2=\frac{\partial u_2}{\partial t} \tag{1.2}$$

$$U=\begin{Vmatrix} v_1\\ v_2\\ p_1\\ p_2\end{Vmatrix},\quad A(U)=\begin{Vmatrix} 0 & 0 & A_{12} & A_{12}\\ 0 & 0 & A_{21} & A_{22}\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{Vmatrix},\quad A_{ij}=-\rho_0^{-1}\frac{\partial\Sigma}{\partial p_i\partial p_j}$$

Let us investigate the solutions of system (1.2) for which $q>0$. We assume that there exist functions Σ', Σ'', Σ''', $(\Sigma'=d\Sigma/dI)$, and that $\Sigma'>0$, $\Sigma''>0$ is for $I\in(3,\infty)$. If $a=2\rho_0^{-1}\Sigma'$, $b=2\rho_0^{-1}(\Sigma'+2q^2\Sigma'')$, then the eigenvalues of the matrix A are

$$\lambda_1=-\sqrt{b},\quad \lambda_2=-\sqrt{a},\quad \lambda_3=\sqrt{a},\quad \lambda_4=\sqrt{b} \tag{1.3}$$

From this follows that system (1.2) is hyperbolic in the narrow sense. We denote by $l_i, r^i, i=1,2,3,4$, respectively, the left and right-hand sides of eigenvectors for which

$$l_{1,4}=\frac{p_1}{q\sqrt{1+b}}\left(1,\frac{p_2}{p_1},\pm\sqrt{b},\ \pm\frac{p_2}{p_1}\sqrt{b}\right) \tag{1.4}$$

$$l_{2,3}=\frac{p_2}{q\sqrt{1+a}}\left(1,-\frac{p_1}{p_2},\pm\sqrt{a},\ \pm\frac{p_1\sqrt{a}}{p_2}\right)$$

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$$r^{1,4}=\frac{p_1}{q\sqrt{1+b}}\begin{Vmatrix}\pm\sqrt{b}\\ \pm\frac{p_2}{p_1}\sqrt{b}\\ 1\\ \frac{p_2}{p_1}\end{Vmatrix},\quad r^{2,3}=\frac{p_2}{q\sqrt{1+a}}\begin{Vmatrix}\pm\sqrt{a}\\ \pm\frac{p_1}{p_2}\sqrt{a}\\ 1\\ -\frac{p_1}{p_2}\end{Vmatrix} \tag{1.5}$$

It follows from formulas (1.4) and (1.5) that the left- and right-hand eigenvectors are normalized as follows:

$$l_i l_i = l_i r^i = 1;\quad l_i r^k = 0,\qquad i \neq k,\ i,\ k = 1, 2, 3, 4 \tag{1.6}$$

$$l_1 l_2 = l_1 l_3 = l_4 l_2 = l_4 l_3 = 0;\ l_1 l_4 = \frac{1-b}{1+b},\quad l_2 l_3 = \frac{1-a}{1+a} \tag{1.7}$$

The characteristic field generated by the i-th eigenvalue of the system, hyperbolic in the narrow sense, is called truly nonlinear in the Lax sense /6/, if the derivative λ_i in the direction of the respective right-hand eigenvector r^i does not vanish, i.e.

$$D\lambda_i\,(U,\, r^i\,(U)) \neq 0 \tag{1.8}$$

The system, hyperbolic in the narrow sense, is truly nonlinear if all of its characteristic fields are truly nonlinear. For (1.2) we have

$$D\lambda_{2,3}\,(U,\, r^{2,3}\,(U)) \equiv 0 \tag{1.9}$$

Consequently, the characteristic fields generated by eigenvalues λ_2 and λ_3 are linearly degenerate in the Lax sense and for any function Σ of deformation energy system (1.2) cannot be truly nonlinear. This shows that the method proposed in /2/ is applicable to the case considered here. Not withstanding this, for the limited $\Sigma'' \geqslant 0$ ensuring the true nonlinearity for the first and third characteristic fields, as well as some requirements as to the initial deformation, the solution of (1.2) may over a finite time become irregular (see Theorem 3.1 below). The linear degeneration of λ_2 and λ_3 provides some possibility for constructing a classic solution (Theorem 3.2).

2. Construction of the evolutionary system.

For initial conditions

$$U\,(x,0)=\begin{Vmatrix}v_1\,(x,0)\\ v_2\,(x,0)\\ p_1\,(x,0)\\ p_2\,(x,0)\end{Vmatrix}=\begin{Vmatrix}v_{10}\,(x)\\ v_{20}\,(x)\\ p_{10}\,(x)\\ p_{20}\,(x)\end{Vmatrix} \tag{2.1}$$

belonging to class c^2, the solution of system (1.2) also belongs to class c^2 for all x in some fairly small time interval $[0, T]$ /3/. Further consideration will be limited to this band in the space (x, t).

The systems of vectors $\{l_i\}_{1:4}$ and $\{r^i\}$ are linearly independent (this follows directly from (1.4) and (1.5)) and together form a biorthogonal system (1.6), (1.7). This enables us to express the gradient U_x in terms of the basis $\{r^i\}$ as follows:

$$U_x = \sum_{i=1}^{4}(l_i U_x)\, r^i$$

Denoting by $w_i\,(x, t) = l_i U_x$-th component of U_x, we use the representation (1.2) in the form of a system of evolutionary equations with unknown functions w_i

$$\frac{\partial w_i}{\partial t} + \lambda_i\,(U\,(x,t))\,\frac{\partial w_i}{\partial x} = \sum_{k=1}^{4}\sum_{m=1}^{4}\gamma_{ikm}\,(U)\,w_k w_m \tag{2.2}$$

$$\gamma_{iim} = -c_{iim} - c_{imi} + \sum_{\substack{j=1\\ j\neq i}}^{4}\frac{\lambda_i-\lambda_m}{\lambda_j-\lambda_i}\,c_{ijm}\,(l_i l_j),\quad m \neq i$$

$$2\gamma_{ikm} = -\frac{\lambda_k-\lambda_m}{\lambda_k-\lambda_i}\,c_{ikm} - \frac{\lambda_m-\lambda_k}{\lambda_m-\lambda_i}\,c_{ikm},\quad k \neq i,\quad m \neq i$$

$$\gamma_{iii} = -c_{iii}$$
$$c_{ikm} = c_{ikm}(U) = l_i(U)\, DA\,[U, r^m(U)]\, r^k(U)$$

Differentiating the equality $l_i A = \lambda_i l_i$, we obtain

$$Dl_i(U, r^i)\, A(U) + l_i(U)\, DA(U, r^i) = D\lambda_i(U, r^i)\, l_i(U) + \lambda_i D\lambda_i(U, r^i)$$

Multiplying both sides by $r^i(U)$ and taking into account the last of relations (2.2), we obtain

$$D\lambda_i(U, r^i) \equiv c_{iii}, \quad 1 \leqslant i \leqslant 4$$

Then from the fourth of relations (2.2) and from (1.9) follows $\gamma_{222} = \gamma_{333} = 0$. To find the remaining γ_{ikm} it is necessary to determine beforehand the coefficient c_{ikm} using formulas (2.2). After respective calculations we obtain

$$DA(U, r^{1,4}) = \left\| \begin{array}{c|cc} 0 & S(p_2) & Q \\ & Q & S(p_3) \\ \hline 0 & & \end{array} \right\|$$
$$S(p) = \frac{2\Sigma''}{p}(q + 4p + 2p^2 q), \quad Q = 4p_3\Sigma''(2 + q)$$

For the coefficients corresponding to the first and fourth characteristic field, we have

$$c_{111} = c_{444} = c_{114} = c_{441} = c_{141} = c_{414} = c_{411} = 2c$$
$$c = (\Sigma'' + 4q\Sigma'' + 2q^2\Sigma''')(1 + b)^{-3/2}$$

All remaining c_{1ij}, c_{4ij} are zero.
For the other group

$$DA(U, r^{2,3}) = \left\| \begin{array}{c|c} 0 & PR \\ & RP \\ \hline 0 & 0 \end{array} \right\|, \quad P = 8p_2\Sigma'', \quad R = \frac{4\Sigma''}{p_3}(p_3^2 - p_2^2)$$
$$c_{221} = c_{331} = c_{224} = c_{334} = c_{231} = c_{321} = c_{234} = c_{324} = 2d$$
$$c_{212} = c_{313} = c_{213} = c_{312} = c_{242} = c_{343} = c_{243} = c_{342} = 2qd$$
$$d = \frac{\Sigma''}{(1 + a)\sqrt{1 + b}}$$

The remaining c_{2ij}, c_{3ij} are zero.
From the fourth of relations (2.2) follows

$$\gamma_{111} = \gamma_{444} = 2c \tag{2.3}$$

Then system (2.2) is transformed to the form

$$\frac{\partial w_1}{\partial t} + \lambda_1(q(x,t))\frac{\partial w_1}{\partial x} = -2c\left(w_1^2 - \frac{1}{1+b}w_1 w_4\right) \tag{2.4}$$
$$\frac{\partial w_2}{\partial t} + \lambda_2(q(x,t))\frac{\partial w_2}{\partial x} = (\gamma_{212}w_1 + \gamma_{224}w_4)w_2 + \gamma_{213}w_1 w_3$$
$$\frac{\partial w_3}{\partial t} + \lambda_3(q(x,t))\frac{\partial w_3}{\partial x} = (\gamma_{313}w_1 + \gamma_{334}w_4)w_3 + \gamma_{324}w_2 w_4$$
$$\frac{\partial w_4}{\partial t} + \lambda_4(q(x,t))\frac{\partial w_4}{\partial x} = -2c\left(w_4^2 - \frac{1}{1+b}w_1 w_4\right)$$
$$\gamma_{212} = \gamma_{313} = -d\left(1 + q - \frac{\sqrt{b} - \sqrt{a}}{2\sqrt{a}}\,\frac{1-a}{1+a}\right)$$
$$\gamma_{224} = \gamma_{334} = -d\left(1 + q + \frac{\sqrt{b} + \sqrt{a}}{2\sqrt{a}}\,\frac{1-a}{1+a}\right)$$
$$\gamma_{213} = \gamma_{324} = -d\left(\frac{\sqrt{b} + \sqrt{a}}{2\sqrt{a}} + \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}}q\right)$$

Initial conditions of the evolutionary system are written as

$$w_{i0}(x) = w_i(x, 0) = l_i(U_0(x))\, U_0'(x), \; i = 1, 2, 3, 4$$

3. The behavior of derivatives of the solution of problem (1.2) - (2.1). We set $k=4$ for $i=1$ and $k=1$ for $i=4$. Let us consider (2.4) as a system for w_1 and w_4. After integration of its characteristic equations

$$\frac{dx}{dt}=\lambda_i, \quad \frac{dw_i}{dt}=2c\left(w_i^2-\frac{1}{1+b}w_iw_k\right), \quad i=1,4$$

we obtain

$$x_i(\alpha_i,t)=\alpha_i+\int_0^t\lambda_i(U(x(s,\alpha_i)s))\,ds \tag{3.1}$$

$$w_i(\alpha_i,t)\equiv w_i(x(\alpha_i,s),t)=w_k(\alpha_i,t)\left\{\frac{1}{w_{i0}(\alpha_i)}+\int_0^t w_k^{-1}(\alpha_i s)\times\frac{2c}{1+b}[q(x_i(\alpha_i s),s)]\,ds\right\}^{-1}$$

$$w_k(\alpha_i,t)\equiv w_k(x_i(\alpha_i,t),t)=\exp\left\{\int_0^t 2c\,[q(x_i(\alpha_i,s)s)]\,w_k[x_i(\alpha_i,s),s]\,ds\right\}, \quad \alpha_i\in R$$

From the analysis of these formulas follows the following theorem.

Theorem 3.1. Let us assume that solution $U(x,t)$ of problem (1.2) - (2.1) is bounded when $0\leqslant t\leqslant T$, $x\in R$, $q>0$ and $\Sigma''(I)\geqslant 0$, $I\in(3,\infty)$, i.e. the first and fourth characteristic fields are truly nonlinear. If function $w_k(t)=w_k(x_i(\alpha_i,t),t)$ is bounded for $0\leqslant t\leqslant T$, $\alpha_i\in R$ then

a) function $w_i(\alpha_i,t)$ is bounded when $w_i(\alpha_i)>0$

b) when $w_i(\alpha_i)<0$

$$|w_i(\alpha_i)|^{-1}<\int_0^T w_k(\alpha_i,s)\frac{2c}{1+b}\{q[x_i(\alpha_i,s),s]\}\,ds$$

There exists a finite time t_* such that

$$\lim_{t\to t_*}|w_i(\alpha_i,t)|=\infty, \quad t_*<T$$

In case b) we have a gradient catastrophy in the smooth solution of problem (1.2), (2.1) /3/.

We shall now investigate the behavior of w_2 and w_3 under the same assumptions.

Theorem 3.2. Let us assume that functions $w_{i2}(t)\equiv w_{i2}(x_2(\alpha_2,t),t)$ and $w_{i3}(t)=w_{i3}(x_3(\alpha_3,t),t)$, $i=1,4$ are bounded when $0\leqslant t\leqslant T$. Then $w_2(x,t)$ and $w_3(x,t)$ are bounded when $0\leqslant t\leqslant T$, if they are bounded for $t=0$.

Proof. From the second and third of equations (2.4) similarly to (3.1) we have

$$x_i(\alpha_i,t)=\alpha_i+\int_0^t\lambda_i(q(x_i(x_i,s))\,ds \tag{3.2}$$

$$w_i(\alpha_i,t)\equiv w_i(x_i(\alpha_i,t),t)=w_i^{-1}(\alpha_i,t)(w_{i0}(\alpha_i)+$$
$$\int_0^t w_i(\alpha_i s)\,\gamma_{i,i-1,i+1}\{q[x_i(x_i,s),s]\,w_{i-1}[x_i(\alpha_i,s)s]\,w_{i+1}[x_i(\alpha_i,s)s]\,ds\}$$

$$w_i(\alpha_i,t)\equiv w_i[x_i(\alpha_i,t),t]=\exp\left(\int_0^t(\gamma_{i1i}(q(x_i(\alpha_i,s),s))\times\right.$$
$$\left.w_1(x_i(\alpha_i,s),s)+\gamma_{i14}(q(x_i(\alpha_i,s),s)\,w_4(x_i(\alpha_i,s),s))\,ds\right)$$

Denoting

$$z_i(t)=\sup_x|w_i(x,t)|, \quad z_{i0}=\sup_x|w_{i0}(x)|, \quad x\in R, \quad i=2,3 \tag{3.3}$$

from these we obtain the inequalities

$$z_i(t)\leqslant z_{i0}+M\int_0^t z_i(s)\,ds, \quad i=2,3 \tag{3.4}$$

where M is a constant which depends on Σ and its derivatives up to third order, as well as on the estimates for $U(x,t)$, $w_{i2}(t)$, $w_{i3}(t)$ in the interval $[0,T]$.

From (3.4) we obtain the inequality

$$z_2(t)+z_3(t)\leqslant(z_{20}+z_{30})\exp(Mt)\leqslant(z_{20}+z_{30})\exp(MT)$$

This theorem shows that waves which correspond to the second and third characteristic fields, behave nearly as linear and do not generate irregularities.

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